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A dual resonance model solves the Yang–Baxter equation

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Abstract. The duality of dual resonance models is shown to imply that the four-point string correlation function solves the Yang–Baxter equation. A reduction of transfer matrices to A_l symmetry is described by a restriction of the KP τ function to Toda molecules.

In this article duality means the stu duality embodied in the dual resonance model [1] developed in the late 1960s and 1970s as a model which describes hadron scattering processes. The purpose of this article is to supply an argument which clarifies the link among three independent subjects in physics: string models in particle physics, solvable lattice models in statistical physics, and soliton theory. In particular it will be shown that the old duality assures that, for a four-point string correlation function, we can solve the Yang–Baxter equation.

The correspondence between the soliton theory and the string models is rather straightforward [2]. The string correlation functions solve the Hirota bilinear difference equation (HBDE), defined as [3]

$$\begin{aligned}\alpha f(k_1+1,k_2,k_3)f(k_1,k_2+1,k_3+1)+\beta f(k_1,k_2+1,k_3)f(k_1+1,k_2,k_3+1)\\+\gamma f(k_1,k_2,k_3+1)f(k_1+1,k_2+1,k_3)=0\end{aligned} \tag{1}$$

with $\alpha+\beta+\gamma=0$. This single equation is equivalent to the KP-hierarchy in soliton theory [4]. A solution of this equation is called the τ function [5].

The soliton theory and the solvable lattice models, on the other hand, share the common structure of integrability, called the quantum inverse-scattering method [6]. There have been interesting works in which a more direct correlation between these two subjects was discussed [7, 8]. Very recently new light was cast on this connection through the papers [9, 10] in which it was pointed out that the algebraic relation satisfied by the transfer matrix of the solvable lattice model with A_l symmetry is nothing but the HBDE (1). Moreover in [9] the authors showed that the linear Bäcklund transformation of the HBDE [11] generates a series of Bethe ansatz solutions.

These results can be summarized such that the HBDE unifies the three problems under consideration. The same solutions, however, are interpreted quite differently from one another: a correlation function of strings, the τ function of soliton theory and a transfer matrix of the solvable lattice models. From a mathematical point of view it is apparent that the HBDE embodies a very large symmetry which guarantees integrability of systems

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with infinite degrees of freedom irrespective of their physical interpretations. Such a mathematical unification of different models, however, does not mean we understand them from a physics point of view.

The case we are concerned with here, the link between the string models and the solvable lattice models, is most obscure since their relation is indirect. Apart from the fact that the correlation functions of strings and the transfer matrix of solvable models are governed by the same equation, their connection is not manifest at all. I would like to fill this gap by showing that the lattice models, whose Boltzmann weight is given by the four-point correlation function of the string models, are solvable. The proof is achieved by noting that the stu duality of the dual resonance models [1] guarantees that the Yang–Baxter equation can be solved.

To begin with let us briefly reformulate the string correlation functions in a way suitable for our discussion in the following [12, 2]. We consider N external strings interacting with each other through the worldsheet specified by the ground state $|G\rangle$. It is given by

$$F_G(K_1, K_2, \ldots, K_N) := \langle 0|W(K_1, g_1)W(K_2, g_2)\ldots W(K_N, g_N)|G\rangle. \tag{2}$$

Here the jth string is assumed to have momentum $K_j^\mu(z)$ distributed along a path $g_j(z)$ in the worldsheet. The path is assumed to close a contour as the local coordinate z of the worldsheet moves around a circle. The interaction takes place via the vertex operator [13]

$$W(K_j, g_j) = \exp\left[\frac{1}{2\pi}\oint\frac{\mathrm{d}z}{z}K_j^\mu(z)X_+^\mu(g_j(z))\right]\exp\left[\frac{1}{2\pi}\oint\frac{\mathrm{d}z}{z}K_j^\mu(z)X_-^\mu(g_j(z))\right]. \tag{3}$$

Here the string coordinate $X^\mu(z) = X_+^\mu(z) + X_-^\mu(z)$ is defined by the following expansion:

$$\begin{aligned} X_-^\mu(z) &= x^\mu + \sum_{n=1}^{\infty}\frac{a_n^\mu}{\sqrt{n}}z^n \\ X_+^\mu(z) &= \mathrm{i}p^\mu \ln z + \sum_{n=1}^{\infty}\frac{a_n^{\dagger\mu}}{\sqrt{n}}z^{-n} \end{aligned} \tag{4}$$

whose components satisfy

$$[x^\mu, p^\nu] = \mathrm{i}\delta^{\mu\nu} \qquad [a_m^\mu, a_n^{\nu\dagger}] = \delta^{\mu\nu}\delta_{mn} \qquad m, n \in \mathbb{Z}_{\geqslant 1}. \tag{5}$$

If K_j^μ is a constant vector k_j^μ, the vertex operator $W(K_j, g_j)$ turns to the ordinary vertex operator for the external ground-state particle of momentum k_j^μ

$$V(k_j, z_j) = \mathrm{e}^{\mathrm{i}k_j^\mu X_+^\mu(z_j)}\mathrm{e}^{\mathrm{i}k_j^\mu X_-^\mu(z_j)} \tag{6}$$

where $z_j = g_j(0)$. Therefore k_j^μ is the barycentric momentum of the jth string. To simplify the formulae, the spacetime indices $\mu, \nu, \ldots$ will be suppressed in what follows.

The empty state $|0\rangle$ is defined by

$$p|0\rangle = a_n|0\rangle = 0 \qquad n = 1, 2, \ldots,$$

while the ground state is defined by $|G_h\rangle = G_h(X)|0\rangle$, where [14]

$$G_h(X) = \theta\left(\zeta - \frac{1}{2\pi}\oint \mathrm{d}X(z)\int^z \omega\right)\exp\left[\frac{1}{8\pi^2}\oint \mathrm{d}X(x)\oint \mathrm{d}X(y)\ln\frac{E(x, y)}{x - y}\right]. \tag{7}$$

$\theta, \omega, E(x, y), \zeta$ are the Riemann theta function, the Abel differential, the prime form, and an arbitrary vector, respectively, all defined on the worldsheet of genus h.

The connection of the correlation function (2) with the τ function of the KP hierarchy was shown [2] to follow

$$\frac{F_G(K_1, K_2, \ldots, K_n)}{F_0(K_1, K_2, \ldots, K_n)} = \tau(t). \tag{8}$$

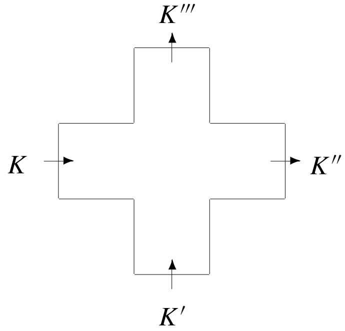

Figure 1.

Here F_0 is given by (2) with $|G\rangle$ replaced by $|0\rangle$, and t denotes the collection of the soliton coordinates $\{t_1, t_2, \ldots\}$ which are related to the string variables by

$$t_n = \frac{1}{n}\sum_{j=1}^{N}\frac{1}{2\pi}\oint\frac{\mathrm{d}z}{z}K_j(z)g_j^n(z) \qquad n = 1, 2, \ldots. \tag{9}$$

Note that, when $K_j^{\mu}(z) = k_j^{\mu}$, this reduces to the Miwa transformation [4]. The proof that (8) satisfies (1) is exactly the same as in [2]. The variables in (1) are any three chosen out of the constant components of $K_j(z)$s.

We now consider the link between the string model and the solvable lattice model. The main step toward this problem is to define the Boltzmann weight properly, so that the Yang–Baxter equation is solved. Here a two-dimensional lattice model is proposed whose links are specified by string momenta $K(z)$s and Boltzmann weight is given by the four-point string correlation function

$$R_{K,K'}^{K'',K'''} = \langle 0|W(K,g)W(K',g')\bar{W}(K'',g'')\bar{W}(K''',g''')|0\rangle. \tag{10}$$

Here $\bar{W}$ is the operator whose in-state and out-state are reversed by changing $K(z)$ to $K(\frac{1}{z})$ in (3). Note that $\bar{W}(K,g)$ can be always replaced by $W(K,\bar{g})$ with $\bar{g}(z) = g(\frac{1}{z})$.

Using this Boltzmann weight the transfer matrix of the model is defined by

$$T_{K_1,K_2,\ldots,K_M}^{K_1',K_2',\ldots,K_M'} = \sum_{\{K_j''\}} R_{K_1'',K_1}^{K_2'',K_1'} R_{K_2'',K_2}^{K_3'',K_2'} \ldots R_{K_M'',K_M}^{K_1'',K_M'}. \tag{11}$$

The summation over K_j'' means the functional integration over all possible paths of strings K_j''. In order to calculate the right-hand side of this formula we use the identity which holds under cyclic permutations

$$\begin{aligned}&\langle 0|W(K_1,g_1)W(K_2,g_2)\ldots W(K_{N-1},g_{N-1})\bar{W}(K_N,g_N)|0\rangle \\ &\qquad = \langle 0|W(K_N,\bar{g}_N)W(K_1,g_1)\ldots W(K_{N-2},g_{N-2})\bar{W}(K_{N-1},\bar{g}_{N-1})|0\rangle\end{aligned} \tag{12}$$

and also the factorization rule [12]

$$\begin{aligned}&\langle 0|W(K_1,g_1)W(K_2,g_2)\ldots W(K_j,g_j)\ldots W(K_{N-1},g_{N-1})\bar{W}(K_N,g_N)|0\rangle \\ &\qquad = \sum_K \langle 0|W(K_1,g_1)W(K_2,g_2)\ldots W(K_j,g_j)\bar{W}(K,g)|0\rangle \\ &\qquad\quad \times\langle 0|W(K,g)W(K_{j+1},g_{j+1})\ldots W(K_{N-1},g_{N-1})\bar{W}(K_N,g_N)|0\rangle.\end{aligned} \tag{13}$$

The summation over K_1'' in (11) reproduces exactly the one-loop string amplitude [15] and all the external ground-state particles are generalized to strings. Hence it turns out that it is given explicitly by the $2M$-point string correlation function, defined on a torus worldsheet associated with the ground state $|G_1\rangle$:

$$\begin{aligned} T^{K'_1,K'_2,\ldots,K'_M}_{K_1,K_2,\ldots,K_M} &= \langle 0|W(K_1,g_1)\bar{W}(K'_1,g'_1)W(K_2,g_2)\ldots\bar{W}(K'_M,g'_M)|G_1\rangle \\ &= F_{G_1}(K_1,\bar{K}'_1,K_2,\bar{K}'_2,\ldots,K_M,\bar{K}'_M). \end{aligned} \tag{14}$$

where $\bar{K}_j$ means that the string K_j is in the outgoing state.

We now proceed to clarify the correspondence of the string correlation functions to the Yang–Baxter equation. Let $|\Phi)_j$ denote the vector on which the external string specified by K_j acts. This is different from the state $|\cdot\rangle$ on which the internal string line X of (4) acts. Every external string belongs to its own vector space [16, 13]. The Boltzmann weight $R^{K_3,K_4}_{K_1,K_2}$ changes the state $|\Phi)_1\otimes|\Phi)_2$ to $|\Phi)_4\otimes|\Phi)_3$. If K_j and K'_j belong to the same vector space and K_k and K'_k belong to another vector space, respectively, $R^{K'_j,K'_k}_{K_j,K_k}$ exchanges the strings. Namely

$$R^{K'_j,K'_k}_{K_j,K_k}|\Phi)_j|\Phi)_k = |\Phi')_k|\Phi')_j.$$

The spectral parameter u which characterizes the Boltzmann weight can be identified by the logarithm of the ratio of $z_j = g_j(0)$ and $z_k = g_k(0)$, $\ln(z_j/z_k)$. Note that they appear as spectral parameters of the inverse scattering problem for the KP hierarchy.

Now writing $R^{K'_j,K'_k}_{K_j,K_k}$ simply as $R_{jk}(u)$ the Yang–Baxter equation is given by

$$R_{12}(u)R_{13}(v)R_{23}(v-u) = R_{23}(v-u)R_{13}(v)R_{12}(u) \tag{15}$$

where $u = \ln z_1 - \ln z_2$ and $v = \ln z_1 - \ln z_3$. This is a sufficient condition for the transfer matrices with different spectral parameters to commute each other, hence the model is solvable [17].

Instead of dealing with (15) we present a more general relation which holds among arbitrary Boltzmann functions.

$$\sum_{K,K',K''} R^{K_4,K_5}_{K,K'}R^{K,K_6}_{K_1,K''}R^{K',K''}_{K_2,K_3} = \sum_{K,K',K''} R^{K_5,K_6}_{K',K}R^{K_4,K}_{K'',K_3}R^{K'',K'}_{K_1,K_2}. \tag{16}$$

The left- and right-hand sides of this equation correspond to figure 2.

To prove (16), I claim that it is nothing more than the duality relation. In fact, by using the permutation symmetry and the factorization property again, the left-hand side of (16)

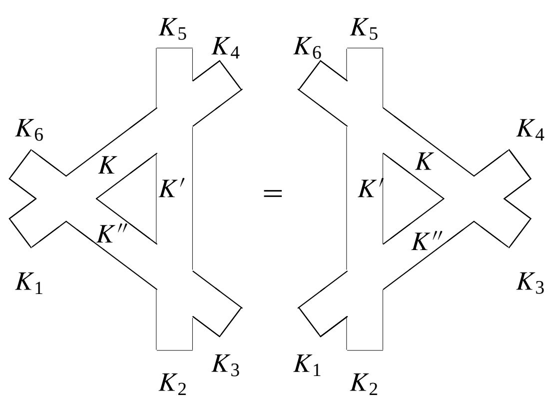

Figure 2.

can be calculated as

$$\begin{aligned}
&\sum_{K,K',K''} \langle 0|W(K,g)W(K',g')\bar{W}(K_4,g_4)\bar{W}(K_5,g_5)|0\rangle \\
&\qquad \times\langle 0|W(K_1,g_1)W(K'',g'')\bar{W}(K,g)\bar{W}(K_6,g_6)|0\rangle \\
&\qquad \times\langle 0|W(K_2,g_2)W(K_3,g_3)\bar{W}(K',g')\bar{W}(K'',g'')|0\rangle \\
&\quad = \sum_{K,K',K''} \langle 0|W(K',g')\bar{W}(K_4,g_4)\bar{W}(K_5,g_5)\bar{W}(K,\bar{g})|0\rangle \\
&\qquad \times\langle 0|W(K,\bar{g})\bar{W}(K_6,g_6)W(K_1,g_1)\bar{W}(K'',\bar{g}'')|0\rangle \\
&\qquad \times\langle 0|W(K'',\bar{g}'')W(K_2,g_2)W(K_3,g_3)\bar{W}(K',g')|0\rangle \\
&\quad = \langle 0|\bar{W}(K_4,g_4)\bar{W}(K_5,g_5)\bar{W}(K_6,g_6)W(K_1,g_1)W(K_2,g_2)W(K_3,g_3)|G_1\rangle
\end{aligned} \tag{17}$$

while the right-hand side becomes

$$\begin{aligned}
&\sum_{K,K',K''} \langle 0|W(K',g')W(K,g)\bar{W}(K_5,g_5)\bar{W}(K_6,g_6)|0\rangle \\
&\qquad \times\langle 0|W(K'',g'')W(K_3,g_3)\bar{W}(K_4,g_4)\bar{W}(K,g)|0\rangle \\
&\qquad \times\langle 0|W(K_1,g_1)W(K_2,g_2)\bar{W}(K'',g'')\bar{W}(K',g')|0\rangle \\
&\quad = \sum_{K,K',K''} \langle 0|W(K,g)\bar{W}(K_5,g_5)\bar{W}(K_6,g_6)\bar{W}(K',\bar{g}')|0\rangle \\
&\qquad \times\langle 0|W(K',\bar{g}')W(K_1,g_1)W(K_2,g_2)\bar{W}(K'',g'')|0\rangle \\
&\qquad \times\langle 0|W(K'',g'')W(K_3,g_3)\bar{W}(K_4,g_4)\bar{W}(K,g)|0\rangle \\
&\quad = \langle 0|\bar{W}(K_5,g_5)\bar{W}(K_6,g_6)W(K_1,g_1)W(K_2,g_2)W(K_3,g_3)\bar{W}(K_4,g_4)|G_1\rangle .
\end{aligned} \tag{18}$$

These are two equivalent expressions of the 6-point one loop amplitude:

$$\begin{aligned}
&F_{G_1}(K_1,K_2,K_3,\bar{K}_4,\bar{K}_5,\bar{K}_6) \\
&\quad = \langle 0|W(K_1,g_1)W(K_2,g_2)W(K_3,g_3)\bar{W}(K_4,g_4)\bar{W}(K_5,g_5)\bar{W}(K_6,g_6)|G_1\rangle .
\end{aligned} \tag{19}$$

In other words they are two different ways of factorizing (19) [12]. The correspondence of (16) to (15) will become explicit if we substitute (K, K', K'') and (K_1', K_2', K_3') for (K_1'', K_2'', K_3'') and (K_4, K_5, K_6), respectively, and let K_j, K_j', K_j'' belong to the same vector space for each $j = 1, 2, 3$. This justifies our claim.

We have just established the direct link between the string models and the solvable lattice models. In the rest of this paper I would like to demonstrate that, through some reduction, we can obtain a familiar solvable lattice model. It will, at the same time, partly explain the mysterious relation between the Yang–Baxter equation and the HBDE recently discussed in [9, 10].

The transfer matrix $T_\nu^{(\mu)}(\lambda)$ of the solvable lattice model associated with A_l symmetry was shown [9, 10] to solve the HBDE (1). The variables μ, ν and λ of $T_\nu^{(\mu)}(\lambda)$ denote the size of the $\mu \times \nu$ rectangular Young tableaux and the spectral parameter which specifies the Boltzmann weight. They are related to the variables k_1, k_2, k_3 of the HBDE according to $\mu = k_2 + k_3 - 1$, $\nu = k_3 + k_1 - 1$, $\lambda = k_1 + k_2 - 1$. This correspondence sounds rather artificial because μ and ν have meaning of the size of Young's tableaux and a range of certain finite intervals, while for the k_1, k_2 and k_3 range all integers or periodic boundary conditions are imposed.

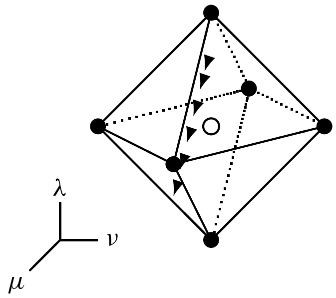

Figure 3.

In order to resolve this unnatural aspect the HBDE is first written in terms of new variables

$$\alpha\tau(\lambda+1,\mu,\nu)\tau(\lambda-1,\mu,\nu)+\beta\tau(\lambda,\mu+1,\nu)\tau(\lambda,\mu-1,\nu) \\ +\gamma\tau(\lambda,\mu,\nu+1)\tau(\lambda,\mu,\nu-1)=0, \tag{20}$$

where $\tau(\lambda,\mu,\nu)=f(k_1,k_2,k_3)$. In the following remark the results used are quoted from recent work [18] in a slightly different form, appropriate to this discussion.

Remark. Let $\tau(\lambda,\mu,\nu)$ be a solution of the HBDE (20), and $\mathbb{A}(\bar\lambda,\bar\mu,\bar\nu)$ an octahedron consisting of the nearest neighbours of the point at $(\lambda,\mu,\nu)=(\bar\lambda,\bar\mu,\bar\nu)$ in the lattice space $\mathbb{Z}^3$, then

$$\bar\tau(\lambda,\mu,\nu)=\begin{cases}\tau(\lambda,\mu,\nu),\ (\lambda,\mu,\nu)\in\mathbb{A}(\bar\lambda,\bar\mu,\bar\nu)\\ 0\quad \text{otherwise}\end{cases} \tag{21}$$

is also a solution to HBDE (20).

This is the smallest piece of Toda lattice which is shown in figure 3. The proof of (21) is simple. Consider another octahedron $\mathbb{A}'$ which shares at least one point of $\mathbb{A}$. Since $\tau(\lambda,\mu,\nu)=0$ on every lattice point surrounding $\mathbb{A}$, the $\tau(\lambda,\mu,\nu)$'s on the octahedron $\mathbb{A}'$ automatically satisfy (20).

The generalization of (21) to an arbitrary size of piece of Toda lattice is straightforward. Let us call such a piece a Toda molecule according to [19]†. Then the smallest unit (21) should be called a Toda atom. A Toda molecule must be rectangular when it is sliced perpendicular to each axis of the lattice, for it to be a solution of the HBDE (20). We can consider a collection of Toda molecules if they are not joined to each other. An example of a slice of such a collection is given in figure 4. Note that each piece can be an independent solution of the HBDE.

Now we go back to our lattice model whose Boltzmann weight is given by $R^{K'',K'''}_{K,K'}$ of (10) but K' and K''' are reduced to their barycentric momenta taking only integral numbers. Using this Boltzmann weight we construct the transfer matrix which carries only integral numbers in its legs. According to the above remarks we can consider any size of Toda molecules in equal basis. From this point of view we can think of this numbering of the legs as specifying the size of the rectangular slice of a molecule instead of the address on the lattice space.

† The term 'Toda molecule' is often used in a slightly different sense. We use this name to mean what we defined in the remark.

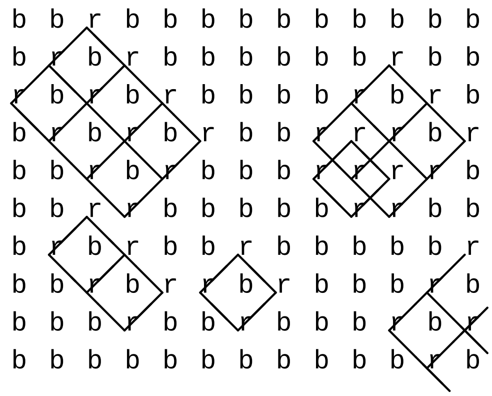

Figure 4. A slice of Toda molecules.

Adopting this convention we consider a solvable lattice model with the A_l symmetry. We identify the transfer matrix associated with the $\mu \times \nu = (k_2 + k_3 - 1) \times (k_3 + k_1 - 1)$ rectangular-type Young's tableaux with the same size of Toda molecule in the (μ, ν) plane. Further identification of the spectral parameter λ with $k_1 + k_2 - 1$ completes the desired correspondence.

In concluding this paper please note the following comments. The partition function of our lattice model itself is a correlation function (2) of strings. Hence it is a solution of the HBDE. This form of a general string correlation function was calculated [20] explicitly to reproduce (2) using the vertex operator W in (3) as a building block. From this point of view the solvable lattice model is nothing but a special case of analogous (or fish net) models [21] discussed in connection with hadron scattering processes. Therefore this model has been shown to be integrable in two folds. Namely, it satisfies the HBDE [2] and also satisfies the Yang–Baxter equation as shown in this paper.

The braid of strings was discussed in [22]. There, the vertex operator W was regarded as representing a state of the string, and a braid of strings was caused through an exchange of the order of Ws. The exchange matrix was derived assuming that states of the strings were not changed under their exchange of order because of duality. Hence it is included as a special case of current work.

Acknowledgments

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References

[1] See for instance, Frampton P 1974 *Dual Resonance Models* (New York: Benjamin)
[2] Saito S 1987 *Phys. Rev. Lett.* **59** 1798
Saito S 1987 *Phys. Rev.* D **36** 1819
Saito S 1988 *Phys. Rev.* D **37** 990
Saito S 1989 *Strings '88* ed S J Gates Jr, C R Preitscopf and W Siegel (Singapore: World Scientific) p 436

Saito S 1991 *Nonlinear Fields: Classical, Random, Semiclassical* ed P Garbaczewski and Z Popowicz (Singapore: World Scientific) p 286
Sogo K 1987 *J. Phys. Soc. Japan* **56** 2291
Carroll R W 1991 *Topics in Soliton Theory (North-Holland Mathematics Studies 167)* (Amsterdam: North-Holland) ch 3
[3] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3787
Hirota R 1983 *Nonlinear Integrable Systems* ed M Jimbo and T Miwa (Singapore: World Scientific) p 17
[4] Miwa T 1982 *Proc. Japan Acad.* **58A** 9
Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Japan* **51** 4116
Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Japan* **51** 4125
[5] Sato M and Sato Y 1980 *Publ. Res. Inst. Math. Sci. (Kyoto University)* **388** 183
Sato M and Sato Y 1981 *Publ. Res. Inst. Math. Sci. (Kyoto University)* **414** 181
Sato M 1981 *Publ. Res. Inst. Math. Sci. (Kyoto University)* **433** 30
Date E, Jimbo M, Kashiwara M and Miwa T 1981 *J. Phys. Soc. Japan* **50** 3806
Date E, Jimbo M, Kashiwara M and Miwa T 1981 *J. Phys. Soc. Japan* **50** 3813
[6] Faddeev L D, Sklyanin E K and Takhtajan L A 1979 *Theor. Math. Phys.* **40** 194
[7] McCoy B, Perk J and Wu T T 1981 *Phys. Rev. Lett.* **46** 757
[8] Korepanov I 1995 *Preprint* solv-int/9506003
Kashaev R M 1995 *Preprint* solv-int/9512004
Korepanov I, Maillard J-M and Sergeev S 1996 *Preprint* PAR-LPTHE 96-53
[9] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1996 Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations *Preprint* hepth/9604080
[10] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5215
Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5267
Kuniba A, Nakamura S and Hirota R 1996 *J. Phys. A: Math. Gen.* **29** 1759
Klümper A and Pearce P 1992 *Physica* **183A** 304
[11] Saito S and Saitoh N 1987 *J. Math. Phys.* **28** 1052
Saitoh N and Saito S 1987 *J. Phys. Soc. Japan* **56** 1664
Saitoh N and Saito S 1990 *J. Phys. A: Math. Gen.* **23** 3017
[12] Saito S 1971 *Phys. Lett.* **34B** 72
Kosterlitz J M and Saito S 1971 *Nucl. Phys.* B **34** 557
Saito S and Kosterlitz J M 1972 *Nucl. Phys.* B **38** 269
[13] Della Selva A and Saito S 1970 *Lett. Nuovo Cimento* **4** 689
[14] Kato H and Saito S 1989 *Lett. Math. Phys.* **18** 117
[15] Frye G and Susskind L 1970 *Phys. Lett.* **31B** 537
Gross D.J, Neveu A, Schrk J and Schwarz J H 1970 *Phys. Rev.* D **2** 697
Shapiro J 1972 *Phys. Rev.* D **5** 1945
[16] Sciuto S 1969 *Lett. Nuovo Cimento* **2** 411
Caneschi L, Schwimmer A and Veneziano G 1969 *Phys. Lett.* **30B** 356
Caneschi L and Schwimmer A 1970 *Lett. Nuovo Cimento* **3** 213
[17] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[18] Saitoh N and Saito S 1996 Coupling of small pieces of Toda lattice and its complex analysis *Int. Symp. Advances in Soliton Theory and its Applications... The 30th Anniversary of the Toda Lattice (Hayama, Japan, December 1–4)*
[19] Hirota R 1987 *J. Phys. Soc. Japan* **56** 4285
[20] Carow-Watamura U, Ezawa Z F and Watamura S 1989 *Nucl. Phys.* B **315** 166
Carow-Watamura U, Ezawa Z F, Tezuka A and Watamura S 1989 *Phys. Lett.* B **221** 299
Carow-Watamura U, Ezawa Z F, Tezuka A and Watamura S 1989 *Nucl. Phys.* B **319** 187
[21] Fairlie D and Nielsen H 1970 *Nucl. Phys.* B **20** 637
Gervais J-L and Sakita B 1971 *Phys. Rev.* D **4** 2291
[22] Cateau H and Saito S 1990 *Phys. Rev. Lett.* **65** 2487